Linear Algebra
Week 1

G-07

26 IX 2024

1 Some Notation

- Sequence of Vectors: $(v_j)_{j=1}^n$
- Set of Vectors: $\{v_j : j \in [n]\}$
- Short way of writing column vectors $\begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} = [v_i]_{i=1}^m$

2 Lengths and Angles

It is a perfectly natural way of thinking to consider vectors as geometric objects that help us move around the space. Therefore we are curious to know how to describe their properties like length (magnitude), position in respect to each other or angels.

2.1 Scalar Product (=Dot Product)

Scalar Product: $v, w \in \mathbb{R}^m : v \cdot w = \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix} = v_1 \cdot w_1 + \cdots + v_m \cdot w_m$

Notation: $< v, w >= \sum_{i=1}^m v_i w_i$

$v \cdot w \in \mathbb{R}$ *Scalar product of two vectors is a real number!*

Observation 1.10. *Let* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^m$ *be vectors and* $\lambda \in \mathbb{R}$ *a scalar. Then*

(i) $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$; (symmetry)

(ii) $(\lambda \mathbf{v}) \cdot \mathbf{w} = \lambda(\mathbf{v} \cdot \mathbf{w}) = \mathbf{v} \cdot (\lambda \mathbf{w})$; (taking out scalars)

(iii) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ *and* $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$; (distributivity)

(iv) $\mathbf{v} \cdot \mathbf{v} \geq 0$, *with equality exactly if* $\mathbf{v} = \mathbf{0}$. (positive-definiteness)

From the lecture notes on scalar product

2.2 Euclidean Norm

We want to know how long our vectors are. In 2 or 3 dimensions this might be a relatively easy task but we need something that also makes sense for higher dimensions.

Definition (Euclidean Norm): $\|x\| := \sqrt{\mathbf{v} \cdot \mathbf{v}}$
In more dimensions:

$$\text{For } v \in \mathbb{R}^m : \|v\| = \sqrt{v_1^2 + v_2^2 + \ldots v_m^2} = \sqrt{\sum_{i=1}^{m} v_i^2}$$

- Unit Vectors: Vectors such that $\|u\| = 1$
- Making Unit Vectors: If you take an arbitrary vector v and divide it with it's norm, you get a unit vector that is parallel to v.

$$\frac{v}{\|v\|} = \frac{1}{\|v\|} \cdot v$$

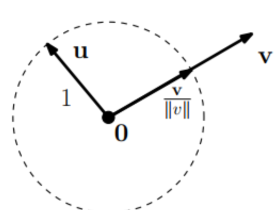

Figure: from the lecture notes on unit vectors

- Standard Unit Vectors: $\mathbb{R}^m : \mathbf{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow$ coordinate i

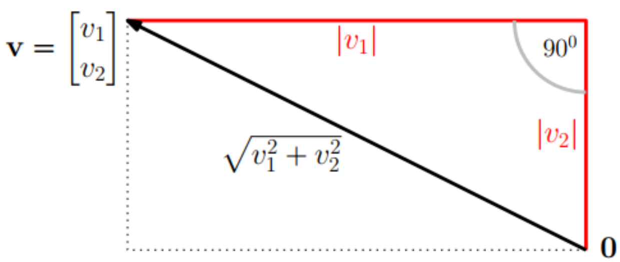

Figure 1.11: The Euclidean norm measures the length of a vector in $\mathbb{R}^2$.

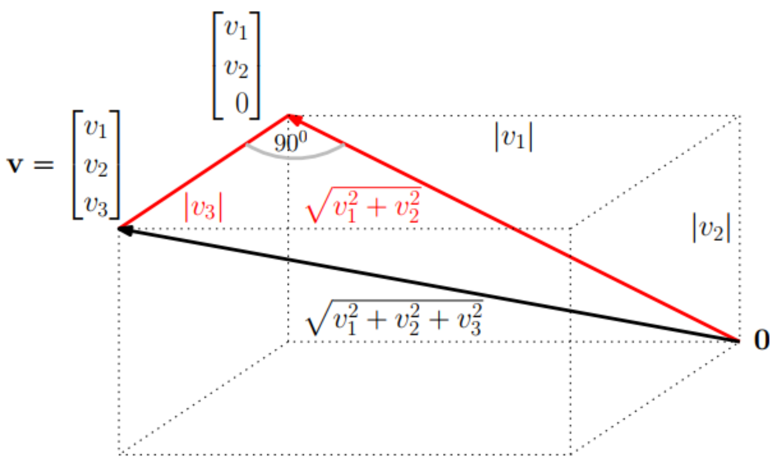

Figure 1.12: The Euclidean norm measures the length of a vector in $\mathbb{R}^3$.

Figure: from the lecture notes on Euclidean Norm

2.3 Cauchy-Schwarz Inequality

Cauchy-Schwarz inequality tells us that the absolute value of the scalar product of two vectors can not be greater than the product of their norms. For any $v, w \in \mathbb{R}^n$:

$$|v \cdot w| \leq \|v\| \cdot \|w\|$$

with equality if and only if the two vectors are collinear.

You can set $v = \lambda \cdot w$ in the inequality above and use the fact that scalars can move (Observation 1.10) along with the fact that $\|\lambda \cdot w\| = |\lambda| \cdot \|w\|$ to convince yourself that the equality holds if the vectors are collinear.

2.4 Angles

You can get help from the Cauchy-Schwarz inequality to determine the angle between two vectors. Again, to come up with a more general method to measure the angle between two vectors is useful for higher dimensions where it might be relatively more difficult to visualize angles in comparison to 2 dimensions.

Let $v, w \in \mathbb{R}^m$ be two nonzero vectors. The angle between them is the unique α between 0 and π (180 degrees) such that:

$$cos(\alpha) = \frac{v \cdot w}{\|v\| \cdot \|w\|} \in [-1, 1]$$

Note that the absolute value of the fraction can not be greater than 1. (Cauchy-Schwarz inequality) This is good, since cosine can only have values in the interval $[-1, 1]$. So this is a well defined formula. You can convince yourself that this relation is true, by reading the corresponding chapter in the lecture notes.

2.5 Perpendicular Vectors

Perpendicular vectors are the vectors that have an angle of 90 degrees between them. A more commonly used term is *orthogonal*. Orthogonality is an important concept in linear algebra and it will have its own section in the lecture notes. An imporatant relation is:

$$u, v \in \mathbb{R}^m \text{ are called orthogonal if } u \cdot v = 0$$

2.6 Triangle Inequality

Let $v, w \in \mathbb{R}^m$. Then

$$\|v + w\| \leq \|v\| + \|w\|$$

which is a generalized version of the triangle inequality we know from tow dimensional triangles that also applies for vectors in higher dimensions.

On proofs of these facts:

Especially when you want to prove the Cauchy-Schwarz inequality or the formula for the angle between two vectors it is helpful to prove it only for unit vectors first. This will make things easier. Afterwards you can show that these facts also apply for vectors that are not unit vectors. See lecture notes for the proofs.

3 Linear Independence

Linear independence is the most important concept in linear algebra and you should definitely develop a good understanding of it. We define linear independence for a collection of vectors.

3.1 Definitions

There are three alternative definitions given in the lecture. They are of course all equivalent, but provide us with different points of view. The first one is as follows:

Linear (in)dependence: Vectors $v_1, v_2, ..., v_n$ are linearly dependent if at least one of them is a linear combination of the others, i.e. there exists an index $k \in [n]$ and scalars λ_j such that

$$v_k = \sum_{\substack{j=1 \\ j \neq k}}^{n} \lambda_i \cdot v_i$$

Otherwise, $v_1, v_2, ..., v_n$ are linearly independent.

Here are the three alternatives for the definition of **linear independence**:

Let $v_1, v_2, \dots v_n \in \mathbb{R}^m$. The following statements are equivalent (meaning that they are either all true, or all false).

(i) None of the vectors is a linear combination of the other ones. (This means, the vectors are linearly independent according to Definition above)

(ii) There are no scalars $\lambda_1, \lambda_2, \dots, \lambda_n$ besides $0, 0, ..., 0$ such that $\sum_{j=1}^{n} \lambda_j \cdot v_j$. We also say that 0 can only be written as a trivial linear combination of the vectors.

(iii) None of the vectors is a linear combination of the previous ones.

Important note (Lemma 1.21. in lecture notes):

A linear combination of linearly independent vectors can be written as a linear combination in only one way.

3.2 Span

Span of a collection of vectors is the set of all linear combinations possible with these vectors.

$$\mathbf{Span}(v_1, v_2, \dots, v_n) := \{\sum_{j=1}^{n} \lambda_j \cdot v_j : \lambda_j \in \mathbb{R} \text{ for all } j \in [n]\}$$

If you consider your steps as vectors, you could move everywhere with a front step, a right step, and an up step -let's imagine you could fly-. You could go into the opposite direction by simply multiplying the vector that represents 'right' with -1. Then these vectors would be enough to take you anywhere you want. Their span would be the whole 3 dimensional space. In contrast if you only had the ability to go right-left and front-back, you would be stuck at your altitude forever and could not go anywhere you want (hiking would be out of the window for sure). The span is a two dimensional plane in 3 dimensional space in this example.

Important Note (Lemma 1.23 in lecture notes): If you add a vector v to a set of vectors $v_1, \dots, v_n$ and if v is already a linear combination of the vectors $v_1, \dots v_n$ then the span stays the same.

You can see this in the example above. While you can move in 2 directions front-back and right-left, the ability to move diagonally would not take you to somewhere new. You could already produce the behavior of the newly added vector with other vectors.

4 Hints

Hints for this week's assignments:

1. See file on the in class exercise on the website.

2. Define the line as $L = \{\lambda \cdot w : \lambda \in \mathbb{R}\}$ Then guess a vector d that yields $w \cdot d = w_1 \cdot d_1 + w_2 \cdot d_2 = 0$ At b) proving that S' is a hyperplane means finding a vector d', for which the result of the multiplication of d' and any vector from S' is equal 0.

3. think of the vectors $1 = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \textit{ and } \begin{bmatrix} \sqrt{1} \\ \sqrt{2} \\ \vdots \\ \sqrt{n} \end{bmatrix}$

4. No hints.

5. Remember the definition of linear (in)dependence - Whith which λ_i's can you have $\sum_{i=1}^{m} \lambda_i \cdot v_i = 0$.

6. We have a formula for this. Use that. Simplify the fraction in a clever way.

7. See the proof of the Fact 1.5 from lecture notes. Make sure that in your calculations the denominator never gets to be 0.

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